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THE NON-AXISYMMETRIC END LOADING OF A TRUNCATED CONE

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Abstract—The problem considered is the equilibrium of a semi-infinite cone, truncated a fixed distance from its apparent apex. The cone is assumed to be loaded on its end surface of truncation, with the ruled sides being free from stress. The present formulation does not require that this loading be axisymmetric.

Total-stress end loading problems are formulated in terms of the three stress variables prescriptible on the truncated end and three auxiliary variables, of the order of stresses with respect to differentiation. The three auxiliary variables carry the displacement information on the end surface and permit the integration of three of the Beltrami equations of compatibility.

The six-vector satisfies a matrix partial differential equation whose constituent equations are obtained from the three integrated Beltrami equations, the defining equations for the auxiliary variables, and the equation of equilibrium containing the variables prescriptible on the truncated surface. The remaining equations of equilibrium are used to determine the stress variables not in the six vector.

A separation of variables of the matrix equation yields a non-selfadjoint matrix differential equation, hence the eigenfunctions are non-orthogonal. A biorthogonality relation is derived from consideration of the adjoint problem to permit the numerical solution of particular boundary value problems.

The decoupling of the axisymmetric problem in the case of axisymmetry is discussed, including the decoupling of the non-axisymmetric biorthogonality into biorthogonalities for the axisymmetric torsion problem and the axisymmetric torsionless problem.

NOTATION

r, θ, ϕ	spherical coordinates
$T_{rr}, T_{r\theta}, T_{r\phi}, T_{\theta\theta}, T_{\theta\phi}, T_{\phi\phi}$	stresses in spherical coordinates
K	first invariant of the stress tensor
∇^2	Laplacian operator
ν, E	material elastic moduli
a	radius of truncation of the cone
β	cone angle for the cone
F, C, D	auxiliary stress variables
u, v, w	displacement components in the r, θ, ϕ directions, respectively
$\omega_{r\theta}, \omega_{r\phi}, \omega_{\theta\phi}$	small rotation vector components
$\{\mathbf{f}\}$	stress six-vector
$[\mathbf{W}_1], [\mathbf{W}_2] \ldots [\mathbf{W}_8]$	matrices in the matrix differential equation
i	imaginary unit
m	index of non-axisymmetry
b_{mn}	nth eigenvalue for the index, m, of non-axisymmetry
$T_{rrmn}, F_{mn}, T_{r\theta mn}, T_{r\phi mn}, C_{mn}, D_{mn}$	the nth eigenstresses for the index, m, of non-axisymmetry
$\{\mathbf{f}_{mn}(\theta)\}$	eigenstress six-vector
$A_{mn}, X_{mn}, Y_{mn}, Z_{mn}$	constants of integration in the eigenstress expansions
x, x_0	$\cos\theta, \cos\beta$
H, I, J, K	constants associated with stresses due to nonself-equilibrated end loadings
$P_\lambda{}^m(\mu)$	Associated Legendre function of first kind of integer order m, complex degree λ, and argument μ
(—)	complex conjugate
$[\ ]^T$	conjugate transpose
$G_{mp1}(\theta), G_{mp2}(\theta), \ldots, G_{mp6}(\theta)$	bi-orthogonal functions
$\{\mathbf{G}_{mp}(\theta)\}$	bi-orthogonal six-vector.

1. INTRODUCTION

This paper presents a formulation in terms of stresses of the non-axisymmetric end loading of a semi-infinite cone, truncated a fixed distance from its apparent apex. The problem considered is a Saint Venant problem since the loaded portion, the truncated end of the cone, represents only a small portion of the surface of the cone and the state of stress of the material decays with respect to the radial component.

The truncated solid cone subjected to self-equilibrated axisymmetric torsionless end loading was studied by Thompson and Little [1], where Papkovich–Neuber displacement potentials were used to obtain series expansions for the stresses in terms of Legendre functions of the first kind. The Fourier coefficients in the series expansions were then evaluated using a least squares technique. The non-axisymmetric end loading of a truncated cone has apparently not yet appeared in the literature.

The exact solution of the three dimensional elasticity problem using a stress formulation, requires the solution of an overdetermined set of nine equations in six unknowns (three Cauchy equations of stress equilibrium and six Beltrami equations of stress compatibility in three normal stresses and three shearing stresses). Because of this inherent difficulty with the stress formulation, in particular with the equations of compatibility, investigators of three dimensional elasticity problems generally prefer the displacement formulation, involving the solution of a determinate set of three equations in three unknown (the three Navier equations of displacement equilibrium in the three components of the displacement vector). However, the method of solution adopted in this work, and in Klemm and Fernandes [2], circumvents the difficulty with the stress formulation through an extension of the method used by Johnson and Little [3] in their investigation of the semi-infinite strip.

In the stress formulation to follow, the three stresses T_{rr}, $T_{r\theta}$ and $T_{r\phi}$ are retained to allow specification of the stress boundary condition on the surface $r = a$ of the cone. Three auxiliary variables F, C and D, which are of the order of stresses with regard to integration, are derived to integrate certain of the Beltrami equations of stress compatibility and to reduce the overdetermined set of nine field equations in six unknowns to a determinate set of six field equations in six unknowns. Additionally, the auxiliary stress variables carry the information of the displacement boundary conditions, and have the ability to represent the homogeneous boundary conditions on the stress free surface.

A separation of variables of the resulting vector partial differential equation yields a non-self-adjoint vector ordinary differential equations which is seen to be an eigenvalue problem. Vector eigensolutions are obtained, the components of which, the eigenstresses of the elasticity problem, each decay at the same rate with respect to the radial coordinate. The eigenvectors, themselves, are not orthogonal, but an adjoint problem is defined which is seen to have the same eigenvalues as the separated field equation and whose vector solutions are pairwise orthogonal to the eigenvectors of the original problem with differing eigenvalues. A biorthogonality relation is derived in this paper to permit the numerical determination of the generalized Fourier coefficients of the vector cone angles.

The non-axisymmetric end loading problem is reduced to the axisymmetric torsion and the axisymmetric torsionless end loading problems, and the nature of the ensuing decoupling between these two problems is studied.

The convergence of the eigenfunction expansions is not studied in the present work, but previously published work on related geometries (e.g. the axisymmetric torsionless loading of the cone in Klemm and Fernandes [2]) show a reasonably rapid convergence for smooth loadings. Similar convergence can be expected for the present problem.

2. THE EIGENFUNCTION EQUATIONS

In the absence of body forces the stress equations of elastostatics, in spherical coordinates ($0 \le r < \infty$, $0 \le \theta < \pi$, $0 \le \phi \le 2\pi$), may be written as three equations of stress equilibrium:

$$r^{-2}\partial(r^2 T_{rr})/\partial r + (r \sin\theta)^{-1}\partial(T_{r\theta}\sin\theta)/\partial\theta - r^{-1}(T_{\theta\theta} + T_{\phi\phi}) + (r\sin\theta)^{-1}\partial T_{r\phi}/\partial\phi = 0 \tag{2.1}$$

$$r^{-3}\partial(r^3 T_{r\theta})/\partial r + (r\sin\theta)^{-1}\partial(T_{\theta\theta}\sin\theta)/\partial\theta + (r\sin\theta)^{-1}\partial T_{\theta\phi}/\partial\phi - r^{-1}T_{\phi\phi}\cot\theta = 0 \tag{2.2}$$

$$r^{-3}\partial(r^3 T_{r\phi})/\partial r + (r\sin^2\theta)^{-1}\partial(T_{\theta\phi}\sin^2\theta)/\partial\theta + (r\sin\theta)^{-1}\partial T_{\phi\phi}/\partial\phi = 0 \tag{2.3}$$

and six equations of stress compatibility:

$$\nabla^2 T_{rr} - 4r^{-2}T_{rr} - 4(r^2 \sin\theta)^{-1}\partial(T_{r\theta}\sin\theta)/\partial\theta - 4(r^2\sin\theta)^{-1}\partial T_{r\phi}/\partial\phi + 2r^{-2}(T_{\theta\theta} + T_{\phi\phi}) + (1+\nu)^{-1}\partial^2 K/\partial r^2 = 0 \quad (2.4)$$

$$\nabla^2 T_{\theta\theta} - 2(r^2\sin^2\theta)^{-1}T_{\theta\theta} + 4r^{-2}\partial T_{r\theta}/\partial\theta - 4(r^2\sin\theta)^{-1}\cot\theta\partial T_{\theta\phi}/\partial\phi + 2r^{-2}T_{rr} + 2r^{-2}T_{\phi\phi}\cot^2\theta + (1+\nu)^{-1}\{r^{-1}\partial K/\partial r + r^{-2}\partial^2 K/\partial\theta^2\} = 0 \quad (2.5)$$

$$\nabla^2 T_{\phi\phi} - 2(r^2\sin^2\theta)^{-1}T_{\phi\phi} + 4r^{-2}T_{r\theta}\cot\theta + 4(r^2\sin\theta)^{-1}\cot\theta\partial T_{\theta\phi}/\partial\phi + 2r^{-2}T_{\theta\theta}\cot^2\theta + 4(r^2\sin\theta)^{-1}T_{r\phi}/\partial\phi + 2r^{-2}T_{rr} + (1+\nu)^{-1}\{r^{-1}\partial K/\partial r + r^{-2}\cot\theta\partial K/\partial\theta + (r^2\sin^2\theta)^{-1}\partial^2 K/\partial\phi^2\} = 0 \quad (2.6)$$

$$\nabla^2 T_{r\theta} - r^{-2}(5+\cot^2\theta)T_{r\theta} - 2(r^2\sin\theta)^{-1}\cot\theta\partial T_{r\phi}/\partial\phi - 2(r^2\sin\theta)^{-1}\partial T_{\theta\phi}/\partial\phi + 2r^{-2}\partial T_{rr}/\partial\theta - 2(r^2\sin\theta)^{-1}\partial(T_{\theta\theta}\sin\theta)/\partial\theta + 2r^{-2}T_{\phi\phi}\cot\theta + (1+\nu)^{-1}\{\partial^2(K/r)/\partial r\partial\theta\} = 0 \quad (2.7)$$

$$\nabla^2 T_{r\phi} - r^{-2}(5+\cot^2\theta)T_{r\phi} - 2(r^2\sin^2\theta)^{-1}\partial(T_{\phi\theta}\sin^2\theta)/\partial\theta + 2(r^2\sin\theta)^{-1}\cot\theta\partial T_{r\theta}/\partial\phi + 2(r^2\sin\theta)^{-1}\partial T_{rr}/\partial\phi - 2(r^2\sin\theta)^{-1}\partial T_{\phi\phi}/\partial\phi + (1+\nu)^{-1}\{\partial^2(K/r\sin\theta)/\partial\phi\partial r\} = 0 \quad (2.8)$$

$$\nabla^2 T_{\theta\phi} - 2r^{-2}(1+2\cot^2\theta)T_{\theta\phi} + 2(r^2\sin\theta)^{-1}\cot\theta\partial T_{\theta\theta}/\partial\phi + 2(r^2\sin\theta)^{-1}\partial T_{r\theta}/\partial\phi + 2r^{-2}\sin\theta\partial(T_{r\phi}/\sin\theta)/\partial\theta - 2(r^2\sin\theta)^{-1}\cot\theta\partial T_{\phi\phi}/\partial\phi + (1+\nu)^{-1}\{r^{-2}\partial^2(K/\sin\theta)/\partial\phi\partial\theta\} = 0 \quad (2.9)$$

where ν is Poisson's ratio, K is the first invariant of the stress tensor

$$K = T_{rr} + T_{\theta\theta} + T_{\phi\phi} \quad (2.10a)$$

and

$$\nabla^2 = r^{-2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + (r^2\sin\theta)^{-1}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + (r\sin\theta)^{-2}\frac{\partial^2}{\partial\phi^2} \quad (2.10b)$$

is the Laplacian in spherical coordinates.

The problem considered is that of an isotropic elastic truncated cone occupying the region

$$\begin{aligned} &0 < a \le r < \infty \\ &0 \le \theta \le \beta \\ &0 \le \phi \le 2\pi. \end{aligned} \quad (2.11)$$

with boundary conditions

$$T_{rr}(a,\theta,\phi) = T^p_{rr}(\theta,\phi) \quad (2.12a)$$

$$T_{r\theta}(a,\theta,\phi) = T^p_{r\theta}(\theta,\phi) \quad (2.12b)$$

$$T_{r\phi}(a,\theta,\phi) = T^p_{r\phi}(\theta,\phi) \quad (2.12c)$$

where T^p_{rr}, $T^p_{r\theta}$ and $T^p_{r\phi}$ are prescribed loading stresses and

$$T_{r\theta}(r,\beta,\phi) = 0 \quad (2.13a)$$

$$T_{\theta\theta}(r,\beta,\phi) = 0 \quad (2.13b)$$

$$T_{\theta\phi}(r,\beta,\phi) = 0. \quad (2.13c)$$

In addition, the regularity condition

$$\text{solution} \to 0 \quad \text{as } r \to \infty \quad (2.14)$$

is assumed to ensure decaying stress solutions.

Three auxiliary stress-type variables F, C and D are defined by the following equations:

$$F = T_{\theta\theta} + T_{\phi\phi} \tag{2.15}$$

$$\partial(rC)\partial r = (\sin\theta)^{-2}\partial[\partial T_{r\theta}/\partial\phi - \partial(T_{r\phi}\sin\theta)/\partial\theta]/\partial\phi - (1-\nu)(1+\nu)^{-1}\partial(F + T_{rr})/\partial\theta \tag{2.16}$$

$$\partial(rD)/\partial r = -\partial[(\sin\theta)^{-1}\partial T_{r\theta}/\partial\phi - (\sin\theta)^{-1}(T_{r\phi}\sin\theta)/\partial\theta]/\partial\theta - (1-\nu)\cdot(1+\nu)^{-1}(\sin\theta)^{-1}\partial(F + T_{rr})/\partial\phi \tag{2.17}$$

where the displacements u, v and w in the r, θ and ϕ coordinate directions, respectively, are related to the auxiliary stress variables through the relations†

$$(1-\nu)rF - 2\nu r T_{rr} = E[2u + (\sin\theta)^{-1}\partial(v\sin\theta)/\partial\theta + (\sin\theta)^{-1}\partial w/\partial\phi] \tag{2.18a}$$

$$2(1+\nu)r(C - T_{r\theta}) = E[2v - 2\partial u/\partial\theta + (\sin\theta)^{-2}\partial\{\partial v/\partial\phi - \partial(w\sin\theta)/\partial\theta\}/\partial\phi] \tag{2.18b}$$

$$2(1+\nu)r(D - T_{r\phi}) = E[2w - 2(\sin\theta)^{-1}\partial u/\partial\phi - \partial\{(\sin\theta)^{-1}\partial v/\partial\phi - (\sin\theta)^{-1}\partial(w\sin\theta)/\partial\theta\}/\partial\theta]. \tag{2.18c}$$

The auxiliary variables F, C and D are used in conjunction with the regularity condition at infinity (2.14) to integrate the Beltrami equations of stress compatibility.

Adding eqns (2.4)–(2.6) yields

$$\nabla^2 K = 0. \tag{2.19}$$

Substituting (2.16) and (2.17) in (2.19), integrating, and setting the function of integration to zero in accordance with the regularity condition (2.14), one has

$$\partial(F + T_{rr})/\partial r - (1+\nu)\cdot(1-\nu)^{-1}(r\sin\theta)^{-1}[\partial(C\sin\theta)/\partial\theta + \partial D/\partial\phi] = 0. \tag{2.20}$$

Similarly, the use of eqns (2.1), (2.2), (2.16) and (2.7) yields

$$r^{-1}\partial(r^2 T_{r\theta})/\partial r + (1+\nu)^{-1}\partial(F - \nu T_{rr})/\partial\theta + C = 0. \tag{2.21}$$

Finally, eqns (2.1), (2.3), (2.17) and (2.8) yield

$$r^{-1}\partial(r^2 T_{r\phi})/\partial r + (1+\nu)^{-1}(\sin\theta)^{-1}\partial(F - \nu T_{rr})/\partial\phi + D = 0. \tag{2.22}$$

Equations (2.1), (2.16), (2.17) and (2.20)–(2.22), which represent a system of six equations in six unknowns, can be represented as the matrix partial differential equation

$$\begin{aligned}
&[\underset{\sim}{\mathbf{W}}_1]\partial\{\mathbf{f}\}/\partial r + [\underset{\sim}{\mathbf{W}}_2]r^{-1}\{\mathbf{f}\} + [\underset{\sim}{\mathbf{W}}_3]r^{-1}\partial\{\mathbf{f}\}/\partial\theta + [\underset{\sim}{\mathbf{W}}_4]r^{-1}\cot\theta\,\partial\{\mathbf{f}\}/\partial\theta \\
&+ [\underset{\sim}{\mathbf{W}}_5](r\sin\theta)^{-1}\partial\{\mathbf{f}\}/\partial\phi + [\underset{\sim}{\mathbf{W}}_6](r\sin\theta)^{-1}\partial^2\{\mathbf{f}\}/\partial\theta\partial\phi \\
&+ [\underset{\sim}{\mathbf{W}}_7](r\sin^2\theta)^{-1}\partial^2\{\mathbf{f}\}/\partial\phi^2 + [\underset{\sim}{\mathbf{W}}_8]r^{-1}\partial^2\{\mathbf{f}\}/\partial\theta^2 = 0
\end{aligned} \tag{2.23}$$

†The auxiliary variables C and D are seen to be related to the small rotations through

$$C = E(1+\nu)^{-1}[\omega_{r\theta} - (\sin\theta)^{-1}\partial\omega_{\theta\phi}/\partial\phi]$$
$$D = E(1+\nu)^{-1}[\omega_{r\phi} + \partial\omega_{\theta\phi}/\partial\theta]$$

where the rotations are given by

$$2\omega_{r\theta} = r^{-1}\partial(rv)/\partial r - r^{-1}\partial u/\partial\theta$$
$$2\omega_{r\phi} = r^{-1}\partial(rw)/\partial r - (r\sin\theta)^{-1}\partial u/\partial\phi$$
$$2\omega_{\theta\phi} = (r\sin\theta)^{-1}[\partial(w\sin\theta)/\partial\theta - \partial v/\partial\phi].$$

where

$$\{\mathbf{f}\} = \begin{bmatrix} T_{rr} \\ F \\ T_{r\theta} \\ T_{r\phi} \\ C \\ D \end{bmatrix}, \quad [\mathbf{W}_1] = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$[\mathbf{W}_2] = \begin{bmatrix} 0 & 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & \operatorname{cosec}^2\theta & & 0 & 1 \\ 0 & 0 & 2 & 0 & & 1 & 0 \\ 0 & 0 & 0 & 2 & & 0 & 1 \\ 0 & 0 & 0 & 0 & -\dfrac{(1+\nu)}{(1-\nu)} & \cot\theta & 0 \\ 2 & -1 & \cot\theta & 0 & & 0 & 0 \end{bmatrix},$$

$$[\mathbf{W}_3] = \begin{bmatrix} \dfrac{1-\nu}{1+\nu} & \dfrac{1-\nu}{1+\nu} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \dfrac{-\nu}{1+\nu} & \dfrac{1}{1+\nu} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\dfrac{(1+\nu)}{(1-\nu)} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad [\mathbf{W}_4] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$[\mathbf{W}_5] = \begin{bmatrix} 0 & 0 & 0 & \cot\theta & 0 & 0 \\ \dfrac{1-\nu}{1+\nu} & \dfrac{1-\nu}{1+\nu} & -\cot\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \dfrac{-\nu}{1+\nu} & \dfrac{1}{1+\nu} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\dfrac{(1+\nu)}{(1-\nu)} \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad [\mathbf{W}_6] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\mathbf{W}_7] = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad [\mathbf{W}_8] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

3. SOLUTION OF THE MATRIX EQUATION

The solution of (2.23) is assumed in the form

$$\{\mathbf{f}\} = \sum_m \sum_n A_{mn} r^{-b_{mn}} e^{im\phi} \{\mathbf{f}_{mn}(\theta)\}. \tag{3.1}$$

where the b_{mn} are eigenvalues governing the decay of the stresses into the material, m is the index of ϕ dependence governing the order of non-axisymmetry, and $i = \sqrt{(-1)}$ is the imaginary unit.

Substituting the stress expansion (3.1) into the matrix differential eqn (2.23), one obtains the eigenvalue problem

$$\{[\underset{\sim}{W}_8]d^2/d\theta^2 + ([\underset{\sim}{W}_3] + \cot\theta[\underset{\sim}{W}_4] + im\ (\sin\theta)^{-1}[\underset{\sim}{W}_6])\,d/d\theta$$
$$+ ([\underset{\sim}{W}_2] + im\ (\sin\theta)^{-1}[\underset{\sim}{W}_5] - m^2(\sin\theta)^{-2}[\underset{\sim}{W}_7])\{\mathbf{f}_{mn}(\theta)\} = b_{mn}[\underset{\sim}{W}_1]\{\mathbf{f}_{mn}(\theta)\}. \tag{3.2}$$

The simultaneous solution of the component equations of (3.2) yields the eigenstresses T_{rrmn}, F_{mn}, $T_{r\theta mn}$, C_{mn} and D_{mn}. The original eigenstresses $T_{\theta\theta mn}$, $T_{\phi\phi mn}$ and $T_{\theta\phi mn}$ can then be obtained form the equilibrium eqns (2.1) and (2.3). Representative eigenstresses for the solid cone are then obtained as listed in Appendix I and are seen to involve a linear combination, with coefficients X_{mn}, Y_{mn} and Z_{mn}, of terms composed of Legrendre functions $P_\lambda{}^m(x)$ of the first kind of complex degree, λ, and integer order, m, agreeing with the index, m, of ϕ-dependence. The complete list of eigenfunctions is given in Fernandes[4].

Imposing the boundary conditions (2.13a–c) is equivalent to requiring the following:

$$T_{r\theta mn}(\beta) = 0 \tag{3.3a}$$

$$(b_{mn}-1)[(2m^2-\cos^2\beta)(\sin\beta)^{-1} + \cos\beta\ d/d\theta]T_{r\phi mn}(\theta)|_{\theta=\beta})]$$
$$+ im(1+\nu)^{-1}[b_{mn}(1+\nu-b_{mn}) + \nu(\cot^2\beta - m^2(\sin\beta)^{-2})]F_{mn}(\beta)$$
$$+ im(1+\nu)^{-1}[b_{mn}(\nu b_{mn}-1-\nu) - (\cot^2\beta - m^2(\sin\beta)^{-2})]T_{rrmn}(\beta) = 0 \tag{3.3b}$$

$$-imb_{mn}(\sin\beta)^{-1}C_{mn}(\beta) + [im\ (\sin\beta)^{-2}\cos\beta + im\ (\sin\beta)^{-1}(d/d\theta)$$
$$-(b_{mn}-1)((d/d\theta) - \cot\beta)]T_{r\phi mn}(\theta)|_{\theta=\beta} + im(1+\nu)^{-1}\cot\beta(T_{rrmn}(\beta) - \nu F_{mn}(\beta)) = 0. \tag{3.3c}$$

Equations (3.3b, c) are obtained by using the compatibility eqns (2.8) and (2.9), the equilibrium eqns (2.1)–(2.3), and the boundary conditions (2.13, a–c).

Applying the boundary conditions (3.3a–c) to the eigenstresses for the solid cone yields three homogenous equations in the unknown coefficients X_{mn}, Y_{mn}, and Z_{mn}. A non-trivial solution requires that the determinant of these three equations be zero, giving the condition from which the eigenvalues b_{mn} were calculated. The eigenequation was solved numerically on an IBM 370 digital computer following the method described by Thompson and Little[1], and the first few eigenvalues are presented in Table 1 for various cone angles and various indices of non-axisymmetry.

The initial roots for $m = 0$ and $m = 1$ are $b_{01} = b_{11} = 2$ and $b_{02} = b_{12} = 3$, and correspond, respectively, to the stresses due to a net force applied at the tip of the non-truncated cone and to an applied net moment. For the axisymmetric case represented by $m = 0$, the problem decouples, with the real roots governing the decay of the stresses due to axisymmetric torsion and the complex roots governing the decay of the stresses due to axisymmetric torsionless loading. For the non-axisymmetric cases, $m \geq 1$, the torsion and torsionless problem do not decouple, and the solution requires two sets of decay parameters: a complex series interlaced with a real series.

The stress and displacement solutions due to both self-equilibrated and non-self equilibrated end loads are listed in Fernandes[4].

4. THE ADJOINT PROBLEM

Since the eigenvalue problem (3.2) is non-self-adjoint, the eigenfunctions, themselves, are not orthogonal. Biorthogonal functions can be obtained, however, from the adjoint problem arising from the generalized non-axisymmetric bi-orthogonality relation.

Using the notation

$$(\bar{\ }) = \text{complex conjugate}$$
$$[\ ]^T = \text{conjugate transpose,}$$

and setting

$$J_{np}^{(m)} = (b_{mn} - b_{mp})\int_0^\beta \{\mathbf{G}_{mp}(\theta)\}^T[\underset{\sim}{W}_1]\{\mathbf{f}_{mn}(\theta)\}\,d\theta, \tag{4.1}$$

Table 1.† $\nu = 0.3$

β	m	n	Complex root‡	Real root
15°	0	1	$11.917 + i5.085$	21.149
		2	24.662 6.160	33.671
		3	36.906 6.892	45.899
	1	1	$17.952 + i5.618$	12.362
		2	30.556 6.551	27.107
		3	42.804 7.175	39.590
	2	1	$9.596 + i3.527$	17.355
		2	23.283 6.064	32.533
		3	36.064 6.875	45.174
	3	1	$14.121 + i4.274$	22.013
		2	28.307 6.438	37.712
		3	41.336 7.157	50.543
	4	1	$18.417 + i4.821$	26.518
		2	33.150 6.765	42.713
		3	46.443 7.409	55.756
45°	0	1	$5.037 + i1.352$	8.139
		2	9.261 1.777	12.278
	1	1	$7.070 + i1.574$	5.401
		2	11.235 1.912	10.125
	2	1	$4.500 + i0.907$	7.102
		2	8.896 1.730	11.969
75°	0	1	$3.753 + i0.046$	5.604
		2	6.221 0.570	8.041
	1	1	$4.971 + i0.406$	4.213

†A more extensive table with a greater number of significant figures is given in Fernandes[4].
‡The complex conjugate is also a root.

expanding by use of the differential eqn (3.2), and integrating by parts, one has

$$J_{np}^{(m)} = [\{(1-\nu)(1+\nu)^{-1}\bar{G}_{mp1} - \nu(1+\nu)^{-1}\bar{G}_{mp3}\}T_{rrmn} + \{(1-\nu)(1+\nu)^{-1}\cdot\bar{G}_{mp1} + (1+\nu)^{-1}\bar{G}_{mp3}\}F_{mn}$$
$$+ \{-(1+\nu)(1-\nu)^{-1}\bar{G}_{mp5}\}C_{mn} + \{im\,(\sin\theta)^{-1}\bar{G}_{mp1} - \cot\theta\bar{G}_{mp2} + \mathrm{d}\bar{G}_{mp2}/\mathrm{d}\theta$$
$$- \bar{G}_{mp2}\,\mathrm{d}/\mathrm{d}\theta\}T_{r\phi mn}]_0^\beta \tag{4.2}$$

where the components

$$\{\mathbf{G}_{mp}(\theta)\}^T = \{\bar{G}_{mp1}(\theta), \bar{G}_{mp2}(\theta), \bar{G}_{mp3}(\theta), \bar{G}_{mp4}(\theta), \bar{G}_{mp5}(\theta), \bar{G}_{mp6}(\theta)\} \tag{4.3}$$

of the ajoint vector are obtained as solutions to the equation adjoint to the differential eqn (3.2):

$$\{[\mathbf{W}_8]^T\mathrm{d}^2/\mathrm{d}\theta^2 + [-[\mathbf{W}_3] - \cot\theta\,[\mathbf{W}_4] - im\,(\sin\theta)^{-1}[\mathbf{W}_6]]^T\mathrm{d}/\mathrm{d}\theta$$
$$+ [-\mathrm{d}([\mathbf{W}_3] + \cot\theta\,[\mathbf{W}_4] + im\,(\sin\theta)^{-1}\,[W_6])/\mathrm{d}\theta + [W_2]$$
$$+ im\,(\sin\theta)^{-1}[W_5] - m^2(\sin\theta)^{-2}[\mathbf{W}_7] - \bar{b}_{mp}[W_1]]^T\}\{G_{mp}(\theta)\} = 0 \tag{4.4}$$

and are presented in Fernandes[4].

The right hand side of (4.2) consists of products of four of the eigenstresses with the adjoint components. To obtain the adjoint boundary conditions on the surface $\theta = \beta$, these products must be reduced to products of three of the eigenstresses with the adjoint components. This reduction can be obtained through the use of the boundary conditions (3.3b, c).

From (4.2), defining

$$L = b_{mn}(b_{mn} - 1)J_{np}^{(m)} \tag{4.5}$$

and substituting (4.5), (3.3b) and ($b_{mn} - 1$) times (3.3c) into (4.2), it is seen that (4.2) contains the products of three of the eigenstresses (T_{rrmn}, F_{mn}, and $T_{r\phi mn}$) with expressions involving the adjoint components. Setting the coefficients of the three eigenstresses equal to zero on the boundary $\theta = \beta$ gives the adjoint boundary conditions in the form:

$$
\begin{aligned}
&[\{(1-\nu)(1+\nu)^{-1}\bar{G}_{mp1} - \nu(1+\nu)^{-1}\bar{G}_{mp3}\}b_{mp}(b_{mp}-1) - (1-\nu)^{-1}x^{-1}m^2 \\
&\quad \times (\sin\theta)^{-1}\bar{G}_{mp5}\{(m^2-1)(1-x^2)^{-1} + \nu b_{mp}(b_{mp}-1) - m^{-2}(1-x^2) \\
&\quad \times b_{mp}(b_{mp}-1)(1+\nu-\nu b_{mp})\} - (1+\nu)^{-1}imx^{-1}b_{mp}\bar{G}_{mp2}\{(x^2-m^2)(1-x^2)^{-1} \\
&\quad - b_{mp}(\nu b_{mp} - 1 - \nu)\}]_0^{\beta} = 0
\end{aligned}
\tag{4.6a}
$$

$$
\begin{aligned}
&[\{(1-\nu)(1+\nu)^{-1}\bar{G}_{mp1} + (1+\nu)^{-1}\bar{G}_{mp3}\}b_{mp}(b_{mp}-1) - (1-\nu)^{-1}m^2(x\sin\theta)^{-1} \\
&\quad \times \bar{G}_{mp5}\{\nu(1-m^2)(1-x^2)^{-1} + b_{mp}(b_{mp}-1)(m^{-2}\sin^2\theta(1+\nu-b_{mp})-1)\} \\
&\quad - (1+\nu)^{-1}imx^{-1}b_{mp}\bar{G}_{mp2}\{\nu(m^2-x^2)(1-x^2)^{-1} + b_{mp}(b_{mp}-1-\nu)\}]_0^{\beta} = 0.
\end{aligned}
\tag{4.6b}
$$

$$
\begin{aligned}
&[\{im(\sin\theta)^{-1}\bar{G}_{mp1} - \cot\theta\bar{G}_{mp2} + \mathrm{d}\bar{G}_{mp2}/\mathrm{d}\theta\}b_{mp}(b_{mp}-1) \\
&\quad - 2(1+\nu)(1-\nu)^{-1}imx^{-1}\bar{G}_{mp5}\{(x^2-m^2)(1-x^2)^{-1} - (b_{mp}-1)\} \\
&\quad - b_{mp}(b_{mp}-1)(x^2-2m^2)(x\sin\theta)^{-1}\bar{G}_{mp2}]_0^{\beta} = 0.
\end{aligned}
\tag{4.6c}
$$

Substituting the adjoint boundary conditions (4.6a–c) into (4.1) and (4.2) yields the non-axisymmetric bi-orthogonality in a form similar to that obtained by Flugge and Kelkar [5], in their discussion of non-axisymmetric problems of the cylinder. The biorthogonality, however, is seen to contain products of the eigenvalues and the corresponding eigenfunctions, and since such terms measure the decay of the applied loading into the material, they are not suitable for the prescription on the boundary. As in Klemm and Little [6], the system of eqns (3.2) are used to convert derivatives in the r direction to derivatives in the θ and ϕ directions and thus convert the variables to a form suitable for specification on the surface $r = a$. The non-axisymmetric biorthogonality is now obtained as:

$$
\begin{aligned}
&(b_{mn} - b_{mp})\left[\int_0^{\beta} [\bar{G}_{mp1}\{b_{mn}(b_{mn}-1)C_{mn}\} + \bar{G}_{mp2}\{b_{mn}(b_{mn}-1)D_{mn}\}\right. \\
&\quad + \bar{G}_{mp3}\{(b_{mn}+1)C_{mn} + 2T_{r\theta mn} + (1+\nu)^{-1}(\mathrm{d}/\mathrm{d}\theta)[(b_{mn}+1+\nu)F_{mn} - 3\nu T_{rrmn} \\
&\quad - \nu(\sin\theta)^{-1}imT_{r\phi mn} - \nu(\sin\theta)^{-1}\,\mathrm{d}(\sin T_{r\theta mn})/\mathrm{d}\theta]\} + \bar{G}_{mp4}\{(b_{mn}+1) \\
&\quad \times D_{mn} + 2T_{r\phi mn} + (1+\nu)^{-1}im(\sin\theta)^{-1}[(b_{mn}+1+\nu)F_{mn} - 3\nu T_{rrmn} \\
&\quad - \nu im(\sin\theta)^{-1}T_{r\phi mn} - \nu(\sin\theta)^{-1}\,\mathrm{d}(\sin\theta T_{r\theta mn})/\mathrm{d}\theta]\} \\
&\quad + \{(1+\nu)(1-\nu)^{-1}(1-b_{mn})\}\bar{G}_{mp5}\{im(\sin\theta)^{-1}D_{mn} + (\sin\theta)^{-1}\,\mathrm{d}(\sin\theta C_{mn})/\mathrm{d}\theta\} \\
&\quad + \bar{G}_{mp6}\{[2 + (1-\nu)^{-1}\nu m^2(\sin\theta)^{-2}]\,T_{rrmn} - [b_{mn} + 1 + (1+\nu)^{-1}m^2(\sin\theta)^{-2}] \\
&\quad \times F_{mn} + im(\sin\theta)^{-1}(D_{mn} + 3T_{r\phi mn}) + (\sin\theta)^{-1}(\mathrm{d}/\mathrm{d}\theta)(\sin\theta[3T_{r\theta mn} + C_{mn} \\
&\quad + (1+\nu)^{-1}\,\mathrm{d}(F_{mn} - \nu T_{rrmn})/\mathrm{d}\theta)])\}]\,\mathrm{d}\theta - [[(1+\nu)^{-1}\bar{G}_{mp1}\{(1-\nu)(b_{mp}+1) + \nu m^2(\sin\theta)^{-2}\} \\
&\quad - (1+\nu)^{-1}im(\cos\theta)^{-1}\bar{G}_{mp2}\{(1-2\nu)\cot^2\theta + (1+\nu)^{-1}(\nu^2+\nu-1)m^2(\sin\theta)^2 \\
&\quad + (b_{mp}+2)(1-\nu) - \nu b_{mp}^2 + (1+\nu)^{-1}\nu^2(\sin\theta)^{-1}(\mathrm{d}/\mathrm{d}\theta)\sin\theta(\mathrm{d}/\mathrm{d}\theta)\} \\
&\quad - \nu(1+\nu)^{-1}\{im(\sin\theta)^{-1}\,\mathrm{d}\bar{G}_{mp2}/\mathrm{d}\theta + (b_{mp}+1)\bar{G}_{mp3}\} \\
&\quad - (1-\nu)m^2(\sin\theta\cos\theta)^{-1}\bar{G}_{mp5}\{\nu(b_{mp}-1) + m^{-2}\sin^2\theta(\nu b_{mp}^2 + \nu - b_{mp} - 1) \\
&\quad + \nu^2(1+\nu)^{-1}(1 - m^{-2}\sin\theta(\mathrm{d}/\mathrm{d}\theta)\sin\theta(\mathrm{d}/\mathrm{d}\theta)\}]T_{rrmn} + [(1+\nu)^{-1} \\
&\quad \times \bar{G}_{mp1}\{(1-\nu)(2-b_{mn}-b_{mp}) + m^2(\sin\theta)^{-2}\} - (1+\nu)^{-1}im(\cos\theta)^{-1}\bar{G}_{mp2} \\
&\quad \times \{(1+\nu)^{-1}\nu(\sin\theta)^{-1}(\mathrm{d}/\mathrm{d}\theta)\sin\theta(\mathrm{d}/\mathrm{d}\theta) + (1-b_{mn})(2b_{mn}+b_{mp}+1) \\
&\quad + \nu(2b_{mp}-1) + (\nu-2)\cot^2\theta + (1+\nu)^{-1}(2-\nu^2)m^2(\sin\theta)^{-2}\} - (1+\nu)^{-1} \\
&\quad \times im(\sin\theta)^{-1}\,\mathrm{d}\bar{G}_{mp2}/\mathrm{d}\theta + (1+\nu)^{-1}\bar{G}_{mp3}\{1-\nu-b_{mn}-b_{mp}\} - (1-\nu)^{-1}
\end{aligned}
$$

$$\times m^2(\cos\theta\sin\theta)^{-1}\bar{G}_{mp5}\{(\nu-2)+m^{-2}\sin^2\theta(b_{mp}^2+b_{mp}b_{mn}+b_{mn}^2-2b_{mn}-2b_{mp}+\nu)$$
$$+(1-\nu)^{-1}\nu(1-m^{-2}\sin\theta\,(\mathrm{d}/\mathrm{d}\theta)\sin\theta\,(\mathrm{d}/\mathrm{d}\theta))\}]F_{mn}+[\{(1-\nu)(1+\nu)^{-1}$$
$$\times\bar{G}_{mp1}-\nu(1+\nu)^{-1}\bar{G}_{mp3}\}-(1-\nu)^{-1}m^2(\cos\theta\sin\theta)^{-1}\bar{G}_{mp5}\{\nu+m^{-2}(\sin^2\theta)$$
$$\times(\nu b_{mp}+2\nu-1)\}-(1+\nu)^{-1}im\,(\cos\theta)^{-1}(1-\nu b_{mp}-3\nu)\bar{G}_{mp2}]\times[(\sin\theta)^{-1}$$
$$\times(\mathrm{d}/\mathrm{d}\theta)(T_{r\theta mn}\sin\theta)]+[(1+\nu)^{-1}im\,(\sin\theta)^{-1}\bar{G}_{mp1}\{\nu b_{mp}+b_{mp}+2\}$$
$$+\bar{G}_{mp2}\{(1+\nu)^{-1}(\nu b_{mp}+2b_{mp}+3-\nu)m^2(\sin\theta\cos\theta)^{-1}-2(b_{mp}+1)\cot\theta\}$$
$$+(b_{mp}+1)\,\mathrm{d}\bar{G}_{mp2}/\mathrm{d}\theta+\bar{G}_{mp3}\{-(1+\nu)^{-1}\nu im\,(\sin\theta)^{-1}\}$$
$$+(1-\nu)^{-1}im\,(\sin\theta)^{-1}\bar{G}_{mp5}\{(\nu b_{mp}+2b_{mp}+3)\tan\theta+((2+\nu)m^2-2(1+\nu))$$
$$\times(\sin\theta\cos\theta)^{-1}\}]T_{r\phi mn}+[(1+\nu)^{-1}\nu im\,(\cos\theta)^{-1}\bar{G}_{mp2}-(1-\nu)^{-1}\nu\,(\cos\theta)^{-1}$$
$$\times\sin\theta\bar{G}_{mp5}]\,[(\sin\theta)^{-1}(\mathrm{d}/\mathrm{d}\theta)\,(\sin\theta C_{mn})]+[im\,(\sin\theta)^{-1}\bar{G}_{mp1}$$
$$+\bar{G}_{mp2}\{(1+\nu)^{-1}(2+\nu)m^2(\sin\theta\cos\theta)^{-1}-2\cot\theta\}+\mathrm{d}\bar{G}_{mp2}/\mathrm{d}\theta$$
$$\left.+(1+\nu)^{-1}(2+\nu)im\,(\cos\theta)^{-1}\bar{G}_{mp5}]D_{mn}\Big|_0^\beta\right]=0. \tag{4.7}$$

The Fourier coefficients A_{mn} in the non-axisymmetric stress expansion (3.1) can now be evaluated through the use of the biorthogonality condition (4.7) following the method outlined below.

5. THE SOLUTION OF SPECIFIC BOUNDARY VALUE PROBLEMS

The boundary value problems which can be solved are those of a truncated cone subjected to applied stresses or displacements on the surface $r = a$, with the surface $\theta = \beta$ being stress free. Using the stress formulation, one must first transform all displacement end conditions into stress-type end conditions through eqns (2.18a–c). The prescribed stress vector is therefore given by

$$\{\mathbf{f}^p\}^T=\{T_{rr}^p, F^p, T_{r\theta}^p, T_{r\phi}^p, C^p, D^p\} \tag{5.1}$$

where the superscript p denotes prescribed values. Since the prescribed values of some of the components are unknown on the surface $r = a$, they are replaced by their formal series expansions, following the method of Johnson and Little[2].

The eigenstress vector on the surface $r = a$ is represented by

$$\{\mathbf{f}(a,\theta,\phi)\}=Ha^{-2}\{\mathbf{f}_{01}(\theta)\}+Ia^{-2}\,\mathrm{e}^{i\phi}\{\mathbf{f}_{11}(\theta)\}+Ja^{-3}\,\mathrm{e}^{i\phi}\,\{\mathbf{f}_{02}(\theta)\}$$
$$+Ka^{-3}\{\mathbf{f}_{12}(\theta)\}+\sum_m\sum_n A_{mn}a^{-b_{mn}}\,\mathrm{e}^{im\phi}\,\{\mathbf{f}_{mn}(\theta)\} \tag{5.2}$$

where the A_{mn} are the Fourier coefficients of the eigenstress expansion corresponding to non-axisymmetric self-equilibrated end loadings, and are to be determined. The first four terms in (5.2) are the stresses due to applied non-self-equilibrated end loading, and the constants H, I, J and K are obtained from the conditions of static equilibrium.

Performing a Fourier analysis in the ϕ-direction on the prescribed function, the vector (5.1) is divided into functions of ϕ, each associated with a different index, m, of ϕ dependence. Following the method of Johnson and Little[3], one obtains from the general bi-orthogonality (4.7), for each value of m and k, a single equation in infinitely many unknowns of the form

$$I_{mk}\{\mathbf{f}^p\}=A_{mk}N_{mk}. \tag{5.3}$$

Here I_{mk} is the kth biorthogonality relation for the index, m, of ϕ dependence corresponding to using the adjoint functions associated with the eigenvalue b_{mk}, and N_{mk} is the normalization constant. The set of all these equations for $k = 1, 2, \ldots$ constitutes an infinite set of linear equations in the infinite number of unknowns $A_{m1}, A_{m2}, \ldots$, and may be solved by truncation.

6. THE AXISYMMETRIC PROBLEM

When the general non-axisymmetric end loading problem of the three dimensional truncated cone is reduced to the axisymmetric case, it decouples into a pure axisymmetric torsion and a pure axisymmetric torsionless end problem. The general axisymmetric case is a superposition of these two problems.

The axisymmetric case is subject to the condition that both the field eqns (2.1)–(2.9) and the boundary conditions, eqns (2.12) and (2.13) be independent of ϕ. The system of eqns (3.2) decouple into a set of two equations in the two axisymmetric torsion unknowns, D and $T_{r\phi}$, and a set of four equations in the four axisymmetric torsionless unknowns T_{rr}, F, $T_{r\theta}$ and C. This latter system of equations is studied in Klemm and Fernandes [2], where a direct derivation is given for the axisymmetric torsionless biorthogonality along with representative numerical work.

Alternatively, these problems may be solved by reducing the non-axisymmetric results. The axisymmetric torsion eigenstresses are obtained by multiplying the non-axisymmetric eigenstresses by m and then setting $m = 0$:

$$T_{r\phi 0n} = A_{0n}(b_{0n} - 2)^{-1} \sin\theta\, (\mathrm{d}/\mathrm{d}x)P_{b_{0n}-2}(x) \tag{6.1a}$$

$$D_{0n} = A_{0n} \sin\theta\, (\mathrm{d}/\mathrm{d}x)P_{b_{0n}-2}(x) \tag{6.1b}$$

$$T_{\theta\phi 0n} = -A_{0n}b_{0n}^{-1}[(b_{0n} - 3)P_{b_{0n}-2}(x) - 2(b_{0n} - 2)^{-1}\, (\mathrm{d}/\mathrm{d}x)P_{b_{0n}-3}(x)] \tag{6.1c}$$

where the constant A_{0n} is given by

$$-i[(1+\nu)^{-1}(1-\nu)b_{0n}(2b_{0n} - 1)^{-1}X_{0n} + Z_{0n}] = A_{0n} \tag{6.2}$$

and where $P_\lambda(\mu)$ are the Legendre Functions of the first kind and the argument $x = \cos\theta$. The remaining eigenstresses T_{rr0n}, F_{0n}, $T_{r\theta 0n}$, C_{0n}, $T_{\theta\theta 0n}$, and $T_{\phi\phi 0n}$ are taken as identically zero.

The axisymmetric torsion eigenequation is obtained as

$$(b_{0n} - 3)P_{b_{0n}-2}(x) - 2(b_{0n} - 2)\,(\mathrm{d}/\mathrm{d}x)P_{b_{0n}-3}(x)|_{\theta=\beta} = 0. \tag{6.3}$$

where the first eigenvalue $b_{01} = 3$ corresponds to the stresses due to an applied moment on the surface $r = a$. The eigenvalues for the axisymmetric torsion end problem of the truncated cone are all real, as Purser (see Love, [7]) showed for the analogous problem of the cylinder.

The axisymmetric torsion displacements are obtained from the non-axisymmetric displacements as

$$Eu = 0 \tag{6.4a}$$

$$Ev = 0 \tag{6.4b}$$

$$E_w = Jr^{-3}\left(\frac{2}{3}\right)(1+\nu)\sin\theta + \sum_n A_{0n}r^{-(b_{0n}-1)}2(1+\nu)b_{0n}^{-1}\sin\theta\,(\mathrm{d}/\mathrm{d}x)\,P_{b_{0n}-2}(x). \tag{6.4c}$$

The Fourier coefficients A_{0n} in the stress and displacement expansions can be evaluated through the use of the axisymmetric torsion biorthogonality relation obtained as a reduction of the general biorthogonality (4.7):

Multiplying the non-axisymmetric adjoint stresses by m, and then setting $m = 0$, yields the axisymmetric torsion biorthogonal functions:

$$\bar{G}_{0p2} = i(b_{0p} - 1)^{-1}[(b_{0p} - 3)(2b_{0p} - 5)^{-1}R_{0p} + T_{0p}]\sin^2\theta\,(\mathrm{d}/\mathrm{d}x)P_{b_{0p}-2}(x) \tag{6.5a}$$

$$\bar{G}_{0p4} = i[(b_{0p} - 3)(2b_{0p} - 5)^{-1}R_{0p} + T_{0p}]\sin^2\theta\,(\mathrm{d}/\mathrm{d}x)P_{b_{0p}-2}(x) \tag{6.5b}$$

$$\bar{G}_{0p1} = \bar{G}_{0p3} = \bar{G}_{0p5} = \bar{G}_{0p6} = 0 \tag{6.5c}$$

where adjoint relations for the axisymmetric torsionless problem are required to satisfy

$$T_{0p} = -(b_{0p} - 3)(2b_{0p} - 5)^{-1}R_{0p} \tag{6.6}$$

to eliminate the terms of the adjoint functions singular for $m = 0$. The reduction of the non-axisymmetric biorthogonality to the axisymmetric torsion then yields

$$(b_{0n} - b_{0p}) \int_0^{\beta} [G_{0p2}(\theta) \cdot D_{0n}(\theta) + G_{0p4}(\theta) \cdot T_{r\phi n}(\theta)] \, d\theta = 0 \tag{6.7}$$

and this biorthogonality can be used in the evaluation of the generalized Fourier coefficients in the manner described in Section 5.

For the pure axisymmetric torsionless problem the terms of eqns (3.2) which are singular for $m = 0$ must be set equal to zero. That is, the constants A_{0n} of eqn (6.2) must be identically zero or

$$Z_{0n} = -(1+\nu)^{-1}(1-\nu)b_{0n}(2b_{0n}-1)^{-1}X_{0n}. \tag{6.8}$$

Imposition of the boundary conditions on $\theta = \beta$ gives the eigenequation governing the axisymmetric torsionless problem derived earlier by Thompson and Little[1]. The roots of this equation after the first are all complex, occurring in complex-conjugate pairs. The lowest eigenvalue, $b_{01} = 2$ corresponds to a net axial force, and the Fourier coefficient corresponding to this term may be evaluated by net applied force considerations. A detailed discussion of this case may be found in Klemm and Fernandes[2] and Fernandes[4].

SUMMARY AND CONCLUSIONS

The present work contains a vector stress formulation for a class of elasticity problems in spherical coordinates. As such it demonstrates the use of the Beltrami equations of stress compatibility in the solution of stress loading problems and provides an alternative to the more usual displacement potential formulations of three dimensional elasticity. The methods of Johnson and Little[3] are extended to permit the calculation of the generalized Fourier coefficients of a vector eigenfunction expansion for end loading problems of the truncated cone. The parameters governing the decay of the solution in the radial direction, hence the width of the Saint Venant boundary region for the end loading region, are given for the first few degrees of non-axisymmetry.

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APPENDIX I

The stress and displacement eigenfunctions

The stress eigenfunctions include:

$$\begin{aligned} T_{rr} = {} & Hr^{-2}(1 + x_0 - 2(2-\nu)(1-2\nu)^{-1}x) + Ir^{-2} e^{i\phi} 2(1-x_0)(1+x_0)^{-1}(\sin\beta)^{-1} \sin\theta \{(1+x_0)(1+x_0)^{-1} - (2-\nu)(1-2\nu)^{-1}\} \\ & + Kr^{-3} e^{i\phi} \sin\theta\{(1-2\nu)^{-1}(5-\nu)x(1+x_0)^{-1} - (1+x_0)(1+x)^{-1}\} + \sum_m \sum_n A_{mn} r^{-b_{mn}} e^{im\phi} T_{rrmn}(\theta). \end{aligned} \tag{A1.1}$$

$$\begin{aligned} T_{r\theta} = {} & Hr^{-2} \sin\theta(x-x_0)(1+x)^{-1} + Ir^{-2} e^{i\phi}[(1-x_0)(\sin\beta)^{-1}\{(2x+1)(1+x)^{-1} - x(1+x_0)^{-1}\} \\ & - (1+x)^{-1}\sin\beta] + Kr^{-3} e^{i\phi}\{(1+\nu)(1-2\nu)^{-1}(x_0^2 - x^2)(1+x_0)^{-1} + (x_0 - x)(1+x)^{-1}\} \\ & + \sum_m \sum_n A_{mn} r^{-b_{mn}} e^{im\phi} T_{r\theta mn}(\theta) \end{aligned} \tag{A1.2}$$

$$\begin{aligned} T_{r\phi} = {} & Ir^{-2} i\, e^{i\phi}[(1-x_0)(\sin\beta)^{-1}\{(1+x_0)^{-1} - (1+2x)(1+x)^{-1}\} + (1+x)^{-1}\sin\beta] + Jr^{-3}\sin\theta \\ & + Kr^{-3} i\, e^{i\phi}\{(1-\nu)(1-2\nu)^{-1}(x_0-1)x - x + (1+x_0)(1+x)^{-1}\} + \sum_n \sum_n A_{mn} r^{-b_{mn}} e^{im\phi} T_{r\phi mn}(\theta) \end{aligned} \tag{A1.3}$$

where $x = \cos\theta$, $x_0 = \cos\beta$ and where

$$T_{rrmn} = X_{mn}[2^{-1}(1+\nu)^{-1}(b_{mn}(b_{mn}+1) - 2(1+\nu))(2b_{mn}-3)^{-1}P^{m}_{b_{mn}-1}(x)] + Y_{mn}P^{m}_{b_{mn}-3}(x) \tag{A1.4}$$

$$\begin{aligned} T_{r\theta mn} = {} & X_{mn}[2^{-1}(1+\nu)^{-1}(b_{mn}^2 - 3b_{mn} + 4 - 2\nu)((b_{mn}-2)(2b_{mn}-3))^{-1}\sin\theta\,(\mathrm{d}/\mathrm{d}x) \\ & \times P^{m}_{b_{mn}-1}(x) + (1+\nu)^{-1}(1-\nu)(b_{mn}-m)((2b_{mn}-1)(b_{mn}-2))^{-1}(\sin\theta)^{-1}P^{m}_{b_{mn}}(x)] \\ & + Y_{mn}(b_{mn}-2)^{-1}\sin\theta\,(\mathrm{d}/\mathrm{d}x)P^{m}_{b_{mn}-3}(x) + Z_{mn}(b_{mn}-2)^{-1}(\sin\theta)^{-1}P^{m}_{b_{mn}-2}(x) \end{aligned} \tag{A1.5}$$

$$\begin{aligned} T_{r\phi mn} = {} & X_{mn}[(1+\nu)^{-1}(1-\nu)im^{-1}(b_{mn}-2)^{-1}\sin\theta\,\{b_{mn}P^{m}_{b_{mn}-1}(x) \\ & -(b_{mn}-m)(2b_{mn}-1)^{-1}(\mathrm{d}/\mathrm{d}x)P^{m}_{b_{mn}}(x)\} + 2^{-1}(1+\nu)^{-1}(-b_{mn}^2 + 3b_{mn} - 4 + 2\nu) \\ & \times((b_{mn}-2)(2b_{mn}-3))^{-1}im\,(\sin\theta)^{-1}P^{m}_{b_{mn}-1}(x)] + Y_{mn}[-im(b_{mn}-2)^{-1}(\sin\theta)^{-1} \\ & \times P^{m}_{b_{mn}-3}(x)] + Z_{mn}[-im^{-1}(b_{mn}-2)^{-1}\sin\theta\,(\mathrm{d}/\mathrm{d}x)P^{m}_{b_{mn}-2}(x)] \end{aligned} \tag{A1.6}$$

where $P_\lambda{}^m(\mu)$ is the associated Legrendre function of the first kind of integer order m, complex degree λ and with real argument, μ.